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STABILITY OF A QUARTIC FUNCTIONAL EQUATION IN MULTI BANACH SPACES: HYERS DIRECT METHOD

B. Muneeswaran[1]., R. Veerasivaji[2]., A. Ponmana Selvan[3] and D.Vignesh[4]

[1]Department of Mathematics, Indian Arts and Science College, Karriyandal,
Kondam - 606 802, Thiruvannamalai Dt., Tamil Nadu, India.
[2]Department of Mathematics, Sri Sankara Arts and Science College (Autonomous), Enathur,
Kanchipuram - 631 561, Tamil Nadu, India.
[3,4] Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur - 635 601, Vellore Dt.,
TamilNadu, India.

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ABSTRACT

In this paper, we investigate the Hyers-Ulam Stability of a Quartic Functional Equation of the form

$f(2x + y) + f(2x-y) = 4 f(x + y) + 4 f(x-y) + 24 f(x) - 6 f(y)$

in Multi Banach Space by using Hyer's Direct method.

INTRODUCTION

The theory of stability is an important branch of the qualitative theory of functional equations. The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by S.M. Ulam [19] about seventy seven years ago. Since then, this question has attracted the attention of many researchers. Note that the affirmative solution to this question was given in the next year by D.H. Hyers [8] in 1941. In the year 1950, T. Aoki [1] generalized Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M.Rassias [17] for linear mappings by considering an unbounded Cauchy difference. In 1994, a further generalization of Th.M. Rassias theorem was obtained by P.Gavruta [7]. After then, the stability problem of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([2, 3, 9, 10, 13, 15, 16, 18, 21, 22]).

In this paper, we prove the Hyers-Ulam stability of a quartic functional equation of the form

$$f(2x + y) + f(2x - y) = 4 f(x + y) + 4 f(x - y) + 24 f(x) - 6 f(y) \tag{1}$$

in Multi Banach space using direct method.

Preliminaries

The multi-Banach space was first investigated by Dales and Polyakov [4]. Theory of multi-Banach spaces is similar to operator sequence space and has some connections with operator spaces and Banach spaces. In 2007 H.G. Dales and M.S. Moslehian [5]first proved the stability of mappings on multi-normed spaces and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces was investigated by

Corresponding author:* **B. Muneeswaran
Department of Mathematics, Indian Arts and Science College, Karriyandal,
Kondam - 606 802, Thiruvannamalai Dt., Tamil Nadu, India.

M.S. Moslehian, K. Nikodem and D. Popa [11] in 2009. In last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians (see, [6, 12, 20]).

Now, we adopt some usual terminology, notion and convension of the theory of multi-Banach spaces from [4, 5]. Let $(X,\|.\|)$ be a complex normed space, and let $k\in\mathbb{N}$. We denote by X^k be the linear space $X\oplus X\oplus\ldots\oplus X$ consisting of k-tuples $(x_1,x_2,\ldots,x_k)$, where, $x_1,x_2,\ldots,x_k\in X$. The linear operations on X^k are defined coordinate-wise. The zero element of either X of X^k is denoted by 0. We denote $\mathbb{N}_k$ by the set of $\{1,2,\ldots,k\}$ and S_k by the group of permutations on k symbols.

Definition 2.1 ***(Multi-norm)*** *A multi-norm on* $\{X^k : k\in\mathbb{N}\}$ *is a sequence* $(\|.\|)=\{\|.\|_k : k\in\mathbb{N}\}$ *such that* $\|.\|_k$ *is a norm on* X^k *for each* $k\in\mathbb{N}$, $\|x_1\|=\|x\|$ *for each* $x\in X$, *and the following axioms are satisfied for each* $k\in\mathbb{N}$ *with* $k\geq 2$*:*

1. $\|(x_{\sigma(1)},\ldots,x_{\sigma(k)})\|_k=\|(x_1\ldots x_k)\|_k$, for $\sigma\in\Psi_k, x_1,\ldots,x_k\in X$;
2. $\|(\alpha_1x_1,\ldots,\alpha_kx_k)\|_k\leq(\max_{i\in\mathbb{N}_k}|\alpha_i|)\|(x_1\ldots x_k)\|_k$ for $\alpha_1\ldots\alpha_k\in\mathbb{C}, x_1,\ldots,x_k\in X$;
3. $\|(x_1,\ldots,x_{k-1},0)\|_k=\|(x_1,\ldots,x_{k-1})\|_{k-1}$, for $x_1,\ldots,x_{k-1}\in X$;
4. $\|(x_1,\ldots,x_{k-1},x_{k-1})\|_k=\|(x_1,\ldots,x_{k-1})\|_{k-1}$ for $x_1,\ldots,x_{k-1}\in X$.

In this case, we say that $((X^k,\|.\|_k):k\in\mathbb{N})$ is a multi - normed space. Suppose that $((X^k,\|.\|_k):k\in\mathbb{N})$ is a multi - normed spaces, and take $k\in\mathbb{N}$. We need the following two properties of multi - norms. They can be found in [4].

$(a)\|(x,\ldots,x)\|_k=\|x\|,\text{for}\quad x\in X,$

$(b)\max_{i\in\mathbb{N}_k}\|x_i\|\leq\|(x_1,\ldots,x_k)\|_k\leq\sum_{i=1}^{k}\|x_i\|\leq k\max_{i\in\mathbb{N}_k}\|x_i\|,\forall x_1,\ldots,x_k\in X.$

It follows from (b) that if $(X,\|.\|)$ is a Banach space, then $(X^k,\|.\|_k)$ is a Banach space for each $k\in\mathbb{N}$; In this case, $((X^k,\|.\|_k):k\in\mathbb{N})$ is a multi - Banach space.

Now, we investigte the Hyers-Ulam stability of the quartic functional equation (1). Throughout this section, let X be a linear space and $\{Y^n:n\in\mathbb{N}\}$ be a multi-Banach space.

Hyers-Ulam Stability Of (1) In Multi-Banach Spaces

Theorem

Let X *be a linear space and* $((Y^n,\|.\|_n):n\in N)$ *be a multi-Banach Spaces. Let* $f:X\to Y$ *is a mapping satisfying* $f(0)=0$ *such that*

$$\sup_{k\in\mathbb{N}}\|(Df(x_1,y_1),\ldots,Df(x_k,y_k))\|_k\leq\varepsilon \tag{2}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k\in X$. Then there exists a unique sextic mapping $S:X\to Y$ such that

$$\sup_{k\in\mathbb{N}}\|(f(x_1)-S(x_1),\ldots,f(x_k)-S(x_k))\|\leq\frac{\varepsilon}{30} \tag{3}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k\in X$.

Proof. Letting $y_i=0$ where $i=1,2,\ldots k$ in (2), we arrive at

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f(2x_1)}{16}-f(x_1),\ldots,\frac{f(2x_k)}{16}-f(x_k)\right)\right\|\leq\frac{\varepsilon}{2.16} \tag{4}$$

Now, Replacing x_i by $2x_i$ where $i=1,2,..,k$ and dividing by 16 in above equation, we get

$$\sup_{k\in N}\left\|\left(\frac{f(2^2x_1)}{16^2}-f(x_1),\ldots,\frac{f(2^2x_k)}{16^2}-f(x_k)\right)\right\|\le\frac{\varepsilon}{2.16^2}+\frac{\varepsilon}{2.16} \tag{5}$$

By using induction for a positive integer $n,$ such that

$$\sup_{k\in N}\left\|\left(\frac{f(2^nx_1)}{16^n}-f(x_1),\ldots,\frac{f(2^nx_k)}{16^n}-f(x_k)\right)\right\|\le\frac{1}{2}\sum_{i=0}^{n-1}\frac{\varepsilon}{16^{i+1}}\le\frac{1}{2}\sum_{i=0}^{\infty}\frac{\varepsilon}{16^{i+1}} \tag{6}$$

Now, we have to show that the sequence $\left\{\frac{f(2^nx)}{16^n}\right\}$ is a Cauchy sequence, we fix $x\in X$ and replacing $x_1,\ldots x_k$ by $x,2x,\ldots,2^{k-1}x$ such that

$$\sup_{k\in N}\left\|\left(\frac{f(2^nx)}{16^n}-\frac{f(2^mx)}{16^m},\ldots,\frac{f(2^{n+k-1}x)}{16^{n+k-1}}-\frac{f(2^{m+k-1}x)}{16^{m+k-1}}\right)\right\|$$

$$\le\sup_{k\in N}\left\|\left(\frac{f(2^nx)}{16^n}-\frac{f(2^mx)}{16^m},\ldots,\frac{1}{16^{k-1}}\left[\frac{f(2^n(2^{k-1}x))}{16^n}-\frac{f(2^m(2^{k-1}x))}{16^m}\right]\right)\right\|$$

Using the definition of Multi-norm, we arrive at

$$\sup_{k\in N}\left\|\left(\frac{f(2^nx)}{16^n}-\frac{f(2^mx)}{16^m},\ldots,\frac{f(2^n(2^{k-1}hx))}{16^n}-\frac{f(2^m(2^{k-1}x))}{16^m}\right)\right\|\le\frac{1}{2}\sum_{i=m}^{n-1}\frac{\varepsilon}{16^{i+1}}. \tag{7}$$

Hence the above inequality (7), shows that $\left\{\frac{f(2^nx)}{16^n}\right\}$ is a Cauchy sequence as $n\to\infty.$ Since Y is complete, then the sequence $\left\{\frac{f(2^nx)}{16^n}\right\}$ converges to a fixed point $S(x)\in Y$ such that

$$S(x)=\lim_{n\to\infty}\frac{f(2^nx)}{16^n}.$$

Therefore, as $n\to\infty$, the inequality (6) implies the inequality (3),

$$\sup_{k\in N}\left\|\left(f(x_1)-S(x_1),\ldots,f(x_k)-S(x_k)\right)\right\|\le\frac{1}{2}\sum_{i=0}^{n-1}\frac{\varepsilon}{16^{i+1}}\le\frac{1}{2}\sum_{i=0}^{\infty}\frac{\varepsilon}{16^{i+1}}\le\frac{\varepsilon}{30}.$$

Obviously, one can find the uniqueness of the mapping $S:X\to Y,$ using the definition of multi-norm. That is, we can easily prove that $S=S'$. This completes the proof.

Corollary 3.2 *Let* X *be a linear space and* $\left((Y^n,\|.\|_n):n\in N\right)$ *be a multi-Banach space. Let* $f:X\to Y$ *be a mapping satisfying* $f(0)=0$ *such that*

$$\sup_{k\in N}\left\|Df(x_1,y_1,\ldots,x_k,y_k)\right\|_k\le\phi(x_1,y_1,\ldots,x_k,y_k) \tag{8}$$

for all $x_1,..,x_k,y_1,..,y_k\in X.$ Then there exists a unique sextic mapping $S:X\to Y$ such that

$$\sup_{k\in N}\left\|f(x_1)-S(x_1),\ldots,f(x_k)-S(x_k)\right\|_k\le\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{16^{i+1}}\phi\left(2^ix_1,x_1,\ldots,2^ix_k,x_k\right) \tag{9}$$

for all $x_1,..,x_k\in X.$

Corollary 3.3 *Let* X *be a linear space and* $\left((Y^n,\|.\|_n):n\in N\right)$ *be a multi-Banach space. Let* $0<p<4$ $,\theta\ge0$ *and* $f:X\to Y$ *be a mapping satisfying* $f(0)=0$ *such that*

$$\sup_{k\in N}\left\|Df(x_1,y_1,\ldots,x_k,y_k)\right\|_k\le\theta\left(\|x_1\|^p+\|y_1\|^p,\ldots,\|x_k\|^p+\|y_k\|^p\right) \tag{10}$$

for all $x_1,..,x_k,y_1,..,y_k\in X.$ Then there exists a unique sextic mapping $S:X\to Y$ such that

$$\sup_{k\in \mathbb{N}}\left\| f(x_1)-S(x_1),\ldots,f(x_k)-S(x_k)\right\|_k \le \frac{\theta}{2.(16-2^p)}\left(\|x_1\|^p,\ldots,\|x_k\|^p\right) \tag{11}$$

for all $x_1,..,x_k \in X$.

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